JV-003-1015001 Seat No. ______

**B. Sc. (Sem. V) (CBCS) (W.I.F. - 2016) Examination**

**October - 2019**

# Math. - 05(A) - Mathematical Analysis - I & Abstract Algebra - I

**Faculty Code : 003**
**Subject Code : 1015001**

Time : **2:30** Hours] [Total Marks : **70**

**1** (a) Answer the following : **4**

(1) Consider the metric space $(R, d)$ where $d$ is the discrete metric and $R$ is the set of all real numbers. Then $N(1, 1.2)$ = ________.

(2) Consider the metric space $(R, d)$ where $d$ is the usual metric on $R$. Write down $N'$ and $\overline{N}$.

(3) Write down the number of limit points of $E = \left\{ r + \frac{1}{n} \middle/ n \in N,\ r = 1, 2, 3, 4 \right\}$ in $(R, d)$ where $d$ is the usual metric on $R$.

(4) If $A = (1, 2)$ is a subset of the metric space $(R, d)$ where $d$ is the usual metric on $R$, write down $\mathring{A}$.

(b) Answer any **one** out of two : **2**

(1) Show that $\frac{1}{4}$ belongs to the Cantor set.

(2) Check whether $d$ defined by $d(x, y) = \left|\sin^2 x - \sin^2 y\right| \forall\, x, y \in R$ is a metric on $R$ or not.

(c) Answer any **one** out of two : **3**

(1) If $(X, d)$ is a metric space, prove that $\left(X, \frac{d}{1+d}\right)$ is also a metric space.

(2) If $(X, d)$ is a metric space and $A, B \subset X$, then prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(d) Answer any **one** out of two : **5**

(1) Let $(X, d)$ be a metric space and $a \in X$. Prove that $N(a, \delta)$ is an open set.

(2) Prove that in a metric space,

(i) an arbitrary union of open sets is open, and

(ii) a finite intersection of open sets is open.

**2** (a) Answer the following : **4**

(1) If $P = \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{9}{10}, 1\right\}$ is a partition of [0, 1], then find $\mu(p)$.

(2) State Darboux's theorem.

(3) Define lower Darboux sum and upper Darboux sum.

(4) If $f$ is a bounded function on $[a, b]$ then write down the relation between $L(P, -f)$ and $U(P, f)$.

(b) Answer any **one** out of two : **2**

(1) If $f(x)=\frac{1}{x}$ where $x \in [1,10]$ and $P=\{1,4,5,10\}$ find $L(P,f)$ and $U(P,f)$.

(2) If $f$ is a bounded function defined on $[a, b]$, then prove that :

$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$, where $P$ is a partition of $[a,b]$ and $m$ and $M$ are the g.l.b. and l.u.b. $f$ in $[a,b]$ respectively.

(c) Answer any **one** out of two : **3**

(1) If $f \in R[a,b]$ then prove that $|f| \in R[a,b]$ and $\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$.

(2) If $f$ is a function defined on $[0, 1]$ by

$$f(x)=\begin{cases} \frac{1}{2^n}, \text{ if } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}; n=0,1,2,3\,..... \\ 0, \text{ if } x=0 \end{cases}$$

then prove that $f \in R[0,1]$ and find $\int_0^1 f\,dx$.

(d) Answer any **one** out of two : **5**

(1) Using definition of Riemann Integral, evaluate $\int_1^3 (2-3x)\,dx$.

(2) If $f, g \in R[a,b]$ then prove that $f+g \in R[a,b]$ and $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

3 (a) Answer the following : **4**

(1) Define primitive of a function.

(2) How many binary operations can be defined on a set A with cardinality 5 ?

(3) Write down the identity element of the General Linear Group *GL* (*n*; *R*).

(4) Consider the group $(R_4, \cdot)$ where $R_4 = \{\pm 1, \pm i\}$.

Write down $O(-i)$ and $O(R_4)$.

(b) Answer any **one** out of two : **2**

(1) Let *G* be a group and $a \in G$ such that $O(a) = n$.

Prove that $O(bab^{-1}) = n \forall b \in G$.

(2) State and prove the first mean value theorem of Integral Calculus.

(c) Answer any **one** out of two : **3**

(1) Prove that $\lim_{n \to \infty} \left( \frac{n^n}{n!} \right)^{\frac{1}{n}} = e$.

(2) Prove that (*G*, *) is a group where $G = R - \{1\}$ and

$a * b = a + b - ab \ \forall a, b \in G$.

(d) Answer any **one** out of two : **5**

(1) State and prove the fundamental theorem of Integral Calculus.

(2) For $0 < \lambda, x < 1$, prove that

$$\sin^{-1} x < \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-\lambda x^2)}} < \frac{\sin^{-1} x}{\sqrt{1-\lambda}}$$

4 (a) Answer the following : **4**

(1) Draw the Lattice diagram of the Klein's group.

(2) Write down the number of generators of the cyclic group $(Z_{20}, +_{20})$.

(3) Check whether $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \in S_4$ is even or odd.

(4) Find the order of permutation :

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix} \in S_5 .$$

(b) Answer any **one** out of two : **2**

(1) If $H$ & $K$ are two subgroups of a group ($G$, *) then check whether $H \cap K$ & $H \cup K$ are subgroups of ($G$, *) or not? Justify your answer.

(2) State and prove Fermat's theorem.

(c) Answer any **one** out of two : **3**

(1) Let $G$ be a group and $a \in G$. Define normalize of $a$ and prove that it is a subgroup of $G$.

(2) Prove that the set of all even permutations of $S_n \, (n \geq 2)$ is a subgroup of $S_n$ of order $\frac{n!}{2}$.

(d) Answer any **one** out of two : **5**

(1) State and prove Lagrange's theorem.

(2) Let $G$ be a group and $H$ be a nonempty subset of $G$. Prove that $H$ is a subgroup of $G$, if and only if $ab^{-1} \in H \; \forall \; a, b \in H$.

5 (a) Answer the following : **4**

(1) State whether the following statement is true or false. Justify your answer.
"If $H$ is a subgroup of $G$, then
$Ha\, Hb = Hab \;\forall\; a, b \in G$."

(2) Define simple group.

(3) Define inner automorphism.

(4) Find the index of the alternating subgroup $A_n$ of the symmetric group $S_n$.

(b) Answer any **one** out of two : **2**

(1) If $H$ is a normal subgroup of $G$, then prove that $xHx^{-1} \subset H \;\forall\; x \in G$.

(2) Let $G$ be a group and let $Z$ be the centre of $G$. Prove that $\varnothing(Z) \subset Z$ where $\varnothing \in a(G)$.

(c) Answer any **one** out of two : **3**

(1) Let $G = (C_0, \cdot)$ and

$$G' = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \Big/ a, b \in R \text{ and } a^2 + b^2 \neq 0 \right\}.$$

$G'$ is a group under matrix multiplication. Prove that $G \cong G'$.

(2) For a fixed element $g$ of a group $G$, if $i_g : G \to G$ defined by $i_g(x) = gxg^{-1} \forall\; x \in G$, then prove that $i_g \in a(G)$.

(d) Answer any **one** out of two : **5**

(1) State and prove Cayley's theorem.

(2) Prove that $(R, +) \cong (R_+, \cdot)$.